A SYMPLECTIC NON-SQUEEZING THEOREM FOR BBM EQUATION

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Abstract. We study the initial value problem for the BBM equation:

$$\begin{cases} u_t + u_x + uu_x - u_{txx} = 0 & x \in \mathbb{T}, t \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

We prove that the BBM equation is globaly well-posed on $H^s(\mathbb{T})$ for $s \geq 0$ and a symplectic non-squeezing theorem on $H^{1/2}(\mathbb{T})$. That is to say the flow-map $u_0 \mapsto u(t)$ that associates to initial data $u_0 \in H^{1/2}(\mathbb{T})$ the solution u cannot send a ball into a symplectic cylinder of smaller width.

1. Introduction

In 1877 Joseph Boussinesq proposed a variety of models for describing the propagation of waves on shallow water surfaces, including what is now refered to as the Korteweg-de Vries (KdV) equation. A scaled KdV equation reads

$$u_t + u_x + \varepsilon(uu_x + u_{xxx}) = 0.$$

The Benjamin-Bona-Mahony (BBM) equation was introduced in [1] as an alternative of the KdV equation. The main argument to derive the BBM equation is that, to the first order in ε, the scaled KdV equation is equivalent to

$$u_t + u_x + \varepsilon(uu_x - u_{txx}) = 0.$$

Indeed, formally we have $u_t + u_x = O(\varepsilon)$, hence $u_{xxx} = -u_{txx} + O(\varepsilon)$.

In this article we shall consider the rescaled BBM equation:

$$u_t + u_x + uu_x - u_{txx} = 0.$$

In 2009, Jerry Bona and Nikolay Tzvetkov proved in [2] that BBM equation is globaly well-posed in $H^s(\mathbb{R})$ if $s \geq 0$, and not even locally well-posed for negative values of s (see also [8]). The result extends to the periodic case (see section 3 below). Let us denote Φ_t the flow map of BBM equation on the circle $\mathbb{T}$. In this article we prove a symplectic non-squeeezing theorem for Φ_t. That is, the flow map cannot squeeze a ball of radius r of $H^{1/2}(\mathbb{T})$ into a symplectic cylinder of radius $r' < r$. Precisely, let $H_0^{1/2}(\mathbb{T}) = \left\{ u \in H^{1/2} / \int_{\mathbb{T}} u = 0 \right\}$ with the Hilbert basis

$$\varphi_n^+(x) = \sqrt{\frac{n}{\pi(n^2+1)}} \cos(nx), \qquad \varphi_n^-(x) = \sqrt{\frac{n}{\pi(n^2+1)}} \sin(nx).$$

Set

$$B_r = \left\{ u \in H_0^{1/2}(\mathbb{T}) \ / \ \|u\|_{H^{1/2}} < r \right\},$$

$$\mathcal{C}_{r,n_0} = \left\{ u = \sum p_n \varphi_n^+ + q_n \varphi_n^- \in H_0^{1/2}(\mathbb{T}) \;/ p_{n_0}^2 + q_{n_0}^2 < r^2 \right\}.$$

The goal of this paper is to prove

Theorem 1.1. *If $\Phi_t(B_r) \subset \mathcal{C}_{R,n_0}$ then $r \leq R$.*

S. Kuksin initiated the investigation of non-squeezing results for infinite dimentional Hamiltonian systems (see [7]). In particular he proved that non-linear wave equation has the non-squeezing property for some nonlinearities. This result were extended to certain stronger nonlinearities by Bourgain [3], and he also proved with a different method that the cubic NLS equation on the circle $\mathbb{T}$ has the non-squeezing property. Using similar ideas Colliander, Keel, Staffilani, Takaoka and Tao obtained the same result for KdV equation on $\mathbb{T}$ (see [4]).

In this article we will use the original theorem of Kuksin. In section 2, we present the construction of a capacity on Hilbert spaces introduced by Kuksin in [7]. This capacity is invariant with respect to the flow of some hamiltonian PDEs provided it has the form "linear evolution + compact". As a corollary of this result we get a non-squeezing theorem for these PDEs. Then we apply this theorem to the BBM equation in section 3. We prove the global wellposedness of BBM equation on $H^s(\mathbb{T})$ for $s \geq 0$, and some estimates on the solutions.

2. Symplectic capacities in Hilbert spaces and non-squeezing theorem

2.1. **The frame work and an abstract non-squeezing theorem.** Let $(Z, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with $\{\varphi_j^\pm / j \geq 1\}$ a Hilbert basis. For $n \in \mathbb{N}$ we denote $Z^n = \mathrm{Span}(\{\varphi_j^\pm / 1 \leq j \leq n\})$, and $\Pi^n : Z \to Z^n$ the corresponding projector. We also denote Z_n the space such that $Z = Z^n \oplus Z_n$. Then, every $z \in Z$ admits the unique decomposition $z = z^n + z_n$ with $z_n \in Z_n$ and $z^n \in Z^n$.

We define $J : Z \to Z$ the skewsymmetric linear operator by

$$J\varphi_j^\pm = \mp \varphi_j^\mp$$

and we supply Z with a symplectic structure with the 2-form ω defined by $\omega(\xi, \eta) = \langle J\xi, \eta \rangle$.

We take a self-adjoint operator A, such that

$$\forall j \in \mathbb{Z},\ A\varphi_j^\pm = \lambda_j \varphi_j^\pm. \tag{1}$$

Define the Hamiltonian

$$f(z) = \frac{1}{2} \langle Az, z \rangle + h(z)$$

where h is a smooth function defined on $Z \times \mathbb{R}$. The corresponding Hamiltonian equation has the form

$$\begin{cases} \dot{z} = JAz + J\nabla h(z) \\ z(0, \cdot) = z_0 \in Z \end{cases} \tag{2}$$

If Z_- is a Hilbert space, we denote

$$Z < Z_-$$

if Z is compactly embedded in Z_- and $\{\varphi_j^\pm\}$ is an orthogonal basis of Z_- (not an orthonormal one!). Clearly Z is dense in Z_-. We identify Z and its dual Z^*. Then $(Z_-)^*$ can be identified with a subspace Z_+ of Z and we have

$$Z_+ < Z < Z_-.$$

Denote $\|\cdot\|_-$ (resp. $\|\cdot\|_+$) the norm of Z_- (resp. Z_+).

We also denote $B_R(Z)$ the ball centered at the origin of radius R.

We impose the following assumptions:

(H1): The equation (2) defines a C^1-smooth global flow map Φ on Z. That is, for all $z_0 \in Z$ the equation (2) has a unique solution $z(t) = \Phi_t(z_0)$ for $t \geq 0$, and the flow map $\Phi_t : z_0 \mapsto z(t)$ is C^1-smooth.

(H2): The flow map Φ is uniformely bounded. That is for each $R > 0$ and $T > 0$, there exists $R' = R'_{R,T}$ such that

$$\Phi_t(B_R(Z)) \subset B_{R'}(Z), \quad \text{for } |t| \leq T.$$

(H3): Writing the flow map $\Phi_t = e^{tJA}(I + \widetilde{\Phi}_t)$, we also impose the following *compactness assumption* : fix $R > 0$ and $T > 0$, there exists $C_{R,T}$ such that

$$\forall u_0, u_0' \in B_R(Z), \ \left\|\widetilde{\Phi}_T(u_0) - \widetilde{\Phi}_T(u_0')\right\|_{Z_+} \leq C_{R,T} \left\|u_0 - u_0'\right\|_Z.$$

Under these assumptions, it is well known that the flow maps Φ_t preserve the symplectic form.

The aim of this section is to show the following non-squeezing theorem

Theorem 2.1. *Assume Φ_T is the flow map of an equation of the form (2) and satisfies the previous assumptions. If Φ_T sends a ball*

$$B_r = \{z \in Z / \|z - \bar{z}\| < r\}, \quad \bar{z} \text{ fixed}$$

into a cylinder

$$\mathcal{C}_{R,j_0} = \left\{ z = \sum p_j \varphi_j^+ + q_j \varphi_j^- \Big/ (p_{j_0} - \bar{p}_{j_0})^2 + (q_{j_0} - \bar{q}_{j_0})^2 < R^2 \right\}_{j_0, \bar{p}_{j_0}, \bar{q}_{j_0} \text{ fixed}}$$

then $r \leq R$.

In fact, this theorem is a simple version of the conservation of a symplectic capacity on Z by the flow map Φ_T (see subsection 2.3.2 below)

Remark 2.2. This theoreme implies the following fact. Fiw $\varepsilon > 0$, a time $T > 0$, a Fourier mode n_0 and $r > 0$ (no smallness conditions are imposed on r or T), then there exists $u_0 \in H^{1/2}(\mathbb{T})$ such that

$$\|u_0\|_{H^{1/2}} < r$$

and

$$|\widehat{u(T)}(n_0)| > \frac{r - \varepsilon}{(n_0^2 + 1)^{1/4}}$$

where u solves (2).

The non-squeezing theorem remains true if we don't suppose that the flow map is global in (H1), but the conclusion would be :
either
$$|\widehat{u(T)}(n_0)| > \frac{r-\varepsilon}{(n_0^2+1)^{1/4}}$$
or
$$\sup_{0\le t\le T} \|u(t)\|_{H^{1/2}} = +\infty.$$
So we impose the global wellposedness in time for (2) in order to rule out the second case.

2.2. **An approximation lemma.** In order to define a capacity, we will need to approximate the flow by finite-dimentional maps. We shall use the following lemma

Lemma 2.3. *Let Φ the flow at time T of an equation (2) satisfying the previous assumptions. For each $\varepsilon > 0$ and $R > 0$, there exists $N \in \mathbb{N}$ such that for $u \in B_R$:*
$$\Phi(u) = e^{tJA}(I+\widetilde{\Phi}_\varepsilon)(I+\widetilde{\Phi}_N)(u) \tag{3}$$
where $(I+\widetilde{\Phi}_\varepsilon)$ and $(I+\widetilde{\Phi}_N)$ are symplectic diffeomorphisms satisfying
$$\|\widetilde{\Phi}_\varepsilon(u)\| \le \varepsilon \qquad \text{for } u \in (I+\widetilde{\Phi}_N)(B_R) \tag{4}$$
$$\left(I+\widetilde{\Phi}_N\right)(u^N+u_N) = \left(I+\widetilde{\Phi}_N\right)(u^N) + u_N \qquad \text{for } u^N \in Z^N, u_N \in Z_N. \tag{5}$$

Proof. Recall that $\Phi = e^{TJA}(I+\widetilde{\Phi})$. First, we observe that for $|t| \le T$, any $R > 0$ and $u, v \in B_R(Z)$ we have
$$\left\|\widetilde{\Phi}(u) - \Pi^N\widetilde{\Phi}(u)\right\|_Z \le \varepsilon_1(N) \underset{N\to+\infty}{\longrightarrow} 0. \tag{6}$$
Indeed, as $K = \bigcup_{|t|\le T} \widetilde{\Phi}(B_r(Z))$ is precompact in Z (by (H3)), then (6) results from the following statement
$$\sup_{u\in K} \left\|u - \Pi^N u\right\| \underset{N\to+\infty}{\longrightarrow} 0.$$
Suppose that the convergence does not hold, then we can find a sequence (u_n) in K such that $\|(I-\Pi^n)u_n\| \ge \varepsilon > 0$. As K is precompact there exists a subsequence (u_{n_j}) such that $u_{n_j} \to u$. For n_j sufficiently large we have
$$\|(I-\Pi^{n_j})(u)\| \le \varepsilon/2, \quad \left\|u_{n_j} - u\right\| \le \varepsilon/2.$$
Hence $\left\|(I-\Pi^{n_j})(u_{n_j})\right\| \le \varepsilon$ and we get a contradiction.

Now we set $h_N = h \circ \Pi^N$. Then $\nabla h_N = \Pi^N \nabla h \Pi^N$. We define Φ^N the time T flow of the equation
$$\dot{v} = J(Av + \nabla h_N(v)) \tag{7}$$
or, equivalently, $v = v^N + v_N \in Z^N + Z_N$ and
$$\begin{cases} \dot{v}^N = J(Av^N + \Pi^N \nabla h(v^N)) \\ \dot{v}_N = JAv_N \end{cases}$$
We write $\Phi^N = e^{TJA}(I+\widetilde{\Phi}_N)$.

Since $\widetilde{\Phi}_N = 0$ outside Z^N, $\widetilde{\Phi}_N$ has the desired form (5). Define

$$\widetilde{\Phi}_\varepsilon = \left(\widetilde{\Phi} - \widetilde{\Phi}_N\right)\left(I + \widetilde{\Phi}_N\right)^{-1},$$

so we have

$$e^{TJA}\left(I + \widetilde{\Phi}_\varepsilon\right)\left(I + \widetilde{\Phi}_N\right) = e^{TJA}\left(I + \widetilde{\Phi}\right) = \Phi.$$

Next we estimate the difference $\widetilde{\Phi} - \widetilde{\Phi}_N$. For $u \in B_R(Z)$ we have

$$\begin{aligned}\left\|\widetilde{\Phi}(u) - \widetilde{\Phi}_N(u)\right\|_Z \le \left\|\widetilde{\Phi}(u) - \Pi^N\widetilde{\Phi}(u)\right\|_Z &+ \left\|\Pi^N\widetilde{\Phi}(u) - \Pi^N\widetilde{\Phi}(\Pi^N u)\right\|_Z \\ &+ \left\|\Pi^N\widetilde{\Phi}(\Pi^N u) - \widetilde{\Phi}_N(u)\right\|_Z.\end{aligned}$$

Hence by (6) and assumption (H3), for $u \in B_R(Z)$ we have

$$\left\|\widetilde{\Phi}(u) - \widetilde{\Phi}_N(u)\right\|_Z \le C\varepsilon(N) \underset{N\to+\infty}{\longrightarrow} 0,$$

so for $u \in \left(I + \widetilde{\Phi}_N\right)(B_R(Z))$

$$\left\|\widetilde{\Phi}_\varepsilon(u)\right\|_Z \le \varepsilon(N) \underset{N\to+\infty}{\longrightarrow} 0.$$

■

2.3. **Symplectic capacities and non-squeezing theorem.**

2.3.1. *Capacities in finite-dimentional space.* Consider $\mathbb{R}^{2n}$ supplied with the standard symplectic structure, that is $\omega(x, y) = \langle Jx, y\rangle$ where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

For $f : \mathbb{R}^{2n} \to \mathbb{R}$ a smooth function we define the hamiltonian vectorfield

$$X_f = J\nabla f.$$

Definition 2.4. Let $\mathcal{O}$ an open set of $\mathbb{R}^{2n}$, $f \in C^\infty(\mathcal{O})$ and $m > 0$. The function f is called *m-admissible* if

- $0 \le f(x) \le m$ for $x \in \mathcal{O}$, and f vanishes on a nonempty open set of $\mathcal{O}$, and $f|_{\partial\mathcal{O}} = m$.
- The set $\{z/f(z) < m\}$ is bounded and the distance from this set to $\partial\mathcal{O}$ is $d(f) > 0$.

Following [6] we define the capacity $c_{2n}(\mathcal{O})$ of an open set $\mathcal{O}$ of $\mathbb{R}^{2n}$ as

$$\begin{aligned}c_{2n}(\mathcal{O}) = \inf\{&m_*/\text{for each } m > m_* \text{ and each } m\text{-admissible function } f \text{ in } \mathcal{O} \\ &\text{the vectorfield } X_f \text{ has a non constant periodic solution of period } \le 1\}.\end{aligned}$$

Theorem 2.5. *c_{2n} is a symplectic capacity, that is*

- *if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $c_{2n}(\mathcal{O}_1) \le c_{2n}(\mathcal{O}_2)$ and if $\varphi : \mathcal{O} \to \mathbb{R}^{2n}$ is a symplectic diffeomorphism then $c_{2n}(\mathcal{O}) = c_{2n}(\varphi(\mathcal{O}))$.*
- *$c_{2n}(\lambda\mathcal{O}) = \lambda^2 c_{2n}(\mathcal{O})$.*

- $c_{2n}(B_1) = c_{2n}(\mathcal{C}_{r,1}) = \pi$ *where*

$$B_r = \left\{(p,q)/\sum(p_j^2+q_j^2) < r^2\right\}, \text{ and } \mathcal{C}_{r,1} = \left\{(p,q)/(p_1^2+q_1^2) < r^2\right\}.$$

See [6] for a proof. An immediate consequence of this theorem is the non-squeezing theorem of M. Gromov [5].

Theorem 2.6. *The ball B_r can be symplecticaly embedded into the cylinder $\mathcal{C}_{R,1}$ if and only if $r \leq R$.*

2.3.2. *Construction of a capacity on Hilbert spaces.* In this section we define a symplectic capacity on Hilbert spaces which is invariant with respect to the flow of the equation (2). We will follow the construction of S. Kuksin (see [7]).

For $\mathcal{O}$ an open set of Z we denote $\mathcal{O}^n = \mathcal{O} \cap Z^n$ and observe that $\partial\mathcal{O}^n \subset \partial\mathcal{O} \cap Z^n$.

Definition 2.7. Let $f \in C^\infty(\mathcal{O})$ and $m > 0$. The function f is called *m-admissible* if

- $0 \leq f(x) \leq m$ for $x \in \mathcal{O}$, and f vanishes on a nonempty open set of $\mathcal{O}$, and $f|_{\partial\mathcal{O}} = m$.
- The set $\{z/f(z) < m\}$ is bounded and the distance from this set to $\partial\mathcal{O}$ is $d(f) > 0$.

Remark 2.8. If f is m-admissible, denoting $\operatorname{supp}(f) = \{z/0 < f(z) < m\}$ we have

$$\operatorname{dist}(f^{-1}(0), \partial\mathcal{O}) \geq d(f),$$
$$\operatorname{dist}(\operatorname{supp}(f), \partial\mathcal{O}) \geq d(f).$$

Denote $f_n = f|_{\mathcal{O}^n}$ and consider X_{f_n} the corresponding hamiltonian vectorfield on $\mathcal{O}^n$.

Definition 2.9. A T-periodic trajectory of X_{f_n} is called *fast* if it is not a stationnary point and $T \leq 1$.

A m-admissible function f is called *fast* if there exists n_0 (depending on f) such that for all $n \geq n_0$ the vectorfield X_{f_n} has a fast solution.

Lemma 2.10. *Each periodic trajectory of X_{f_n} is contained in $\operatorname{supp}(f) \cap Z^n$.*

Proof. Pick $z \in \mathcal{O}^n \backslash \operatorname{supp}(f)$, f_n takes either its minimal or maximal value in z, hence $X_{f_n}(z) = 0$. Therefore z is a stationnary point and a fast trajectory cannot pass through it. ■

We are now in position to define a capacity c.

Definition 2.11. For an open set $\mathcal{O}$ of Z its capacity equals to

$$c(\mathcal{O}) = \inf\{m_*/\text{each } m\text{-admissible function with } m > m_* \text{ is fast}\}.$$

Proposition 2.12. *Assume that $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}$ are open sets of Z and $\lambda \neq 0$*

1. *if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $c(\mathcal{O}_1) \leq c(\mathcal{O}_2)$;*
2. *$c(\lambda\mathcal{O}) = \lambda^2 c(\mathcal{O})$.*

Proof. (1) Assume $m < c(\mathcal{O}_1)$, by definition of c there exists a m-admissible function f of $\mathcal{O}_1$ which is not fast. Hence, there exists a sequence $(n_j) \to +\infty$ such that for every $j \in \mathbb{N}$, $X_{f_{n_j}}$ has no fast periodic trajectory. Define $\widetilde{f}$ on $\mathcal{O}_2$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{O}_1 \\ m & \text{otherwise} \end{cases}$$

The function $\widetilde{f}$ is clearly m-admissible on $\mathcal{O}_2$.
By lemma 2.10, for each $j \in \mathbb{N}$, each fast solution $x(t)$ of $X_{\widetilde{f}_{n_j}}$ lies in $\text{supp}\widetilde{f} \cap Z^{n_j}$ $=\text{supp} f \cap Z^{n_j}$. Hence $x(t)$ is a fast trajectory of $X_{f_{n_j}}$ ($X_{\widetilde{f}_{n_j}}$ and $X_{f_{n_j}}$ are the same vectorfields on $\text{supp}(f)$ by definition of $\text{supp}(f)$). Therefore, for each $j \in \mathbb{N}$ the vectorfield $X_{\widetilde{f}_{n_j}}$ of $\mathcal{O}_2$ has no fast trajectory. Hence $\widetilde{f}$ is m-admissible but is not fast. Thus $c(\mathcal{O}_2) \geq m$, and the first assertion follows.

(2) Define $f^\lambda = \lambda^2 f(\lambda^{-1}\cdot)$ on $\lambda\mathcal{O}$. Clearly f is m-admissible on $\mathcal{O}$ if and only if f^λ is $\lambda^2 m$-admissible on $\lambda\mathcal{O}$. Moreover $z(t) \in \mathcal{O}^n$ is a T-periodic trajectory of X_{f_n} if and only if $\lambda z(t) \in \lambda\mathcal{O}^n$ is a T-periodic trajectory of $X_{f_n^\lambda}$. Therefore $c(\lambda\mathcal{O}) = \lambda^2 c(\mathcal{O})$. ■

Lemma 2.13. *If $F : Z \to Z$ has the form*

$$F(z^n + z_n) = F^n(z^n) + z_n \qquad z = z^n + z_n \in Z = Z^n \oplus Z_n$$

with F^n a symplectic diffeomorphism of Z^n, then $c(\mathcal{O}) = c(F(\mathcal{O}))$, for each open set $\mathcal{O}$ of Z.

Proof. We observe that if f is m-admissible in $F(\mathcal{O})$ and f is fast then $f \circ F$ is m-admissible in $\mathcal{O}$ and $f \circ F$ is fast. Indeed $F^* : f \mapsto f \circ F$ clearly sends m-admissible functions in $F(\mathcal{O})$ to similar ones in $\mathcal{O}$, and for $p \geq n$ it tranforms $X_{(f\circ F)^p}$ into X_{f^p}. Hence admissible and fast functions are preserved by F and its inverse (F is the identity outside of Z^n which is a finite-dimentional space), and the result follows. ■

Proposition 2.14. *For each open set $\mathcal{O}$ of Z and ξ in Z, we have*

$$c(\mathcal{O}) = c(\mathcal{O} + \xi).$$

Proof. Denote $\mathcal{O}_\xi = \mathcal{O} + \xi$. It is sufficient to prove that $c(\mathcal{O}) \leq c(\mathcal{O} + \xi)$ (change ξ into $-\xi$).
Denote $\xi = \xi^{n_0} + \xi_{n_0} \in Z^{n_0} + Z_{n_0}$ (n_0 will be fixed later) and $\mathcal{O}_1 = \mathcal{O} + \xi^{n_0}$.By lemma 2.13 $c(\mathcal{O}_1) = c(\mathcal{O})$. We also remark that $\mathcal{O}_\xi = \mathcal{O}_1 + \xi_{n_0}$.

Take any m-admissible function f on $\mathcal{O}_\xi$ with $m > c(\mathcal{O})$. We wish to check that f is fast.
Since $\partial\mathcal{O}_\xi \subset \partial\mathcal{O}_1 + \xi_{n_0}$ and $\|\xi_n\| \underset{n\to+\infty}{\longrightarrow} 0$, we have

$$\text{dist}(\partial\mathcal{O}_1, \partial\mathcal{O}_\xi) \leq \text{dist}(\partial\mathcal{O}_1, \partial\mathcal{O}_1 + \xi_{n_0}) \leq \|\xi_{n_0}\| \underset{n_0\to+\infty}{\longrightarrow} 0.$$

Pick n_0 such that

$$\text{dist}(\partial\mathcal{O}_1, \partial\mathcal{O}_\xi) \leq \|\xi_{n_0}\| < \frac{1}{2} d(f). \tag{8}$$

We extend f outside O_ξ with $f(z) = m$ if $z \notin \mathcal{O}_\xi$ and we denote $\widetilde{f}$ its restriction to $\mathcal{O}_1$.

f equals m on a $d(f)$-neighbourhood of $\partial\mathcal{O}_\xi$. By (8), we deduce that $\widetilde{f}$ equals m on a $\frac{1}{2}d(f)$-neighbourhood of $\partial\mathcal{O}_1$.

By remark 2.8 we have $\mathrm{dist}(f^{-1}(0), \partial\mathcal{O}_\xi) \geq d(f)$. Hence, by (8),we have $\mathrm{dist}(f^{-1}(0), \partial\mathcal{O}_1) \geq \frac{1}{2}d(f)$, and in particular $\widetilde{f}$ vanishes on a nonempty open set of $\mathcal{O}_1 \cap \mathcal{O}_\xi \subset \mathcal{O}_1$. Therefore $\widetilde{f}$ is m-admissible.

Since $c(\mathcal{O}_1) = c(\mathcal{O}) < m$, it follows that $X_{\widetilde{f}_n}$ has a fast trajectory in $\mathcal{O}_1^n$ if $n \geq n_0$ is sufficiently large. By lemma 2.10 this trajectory lies in $\mathrm{supp}\widetilde{f} = \mathrm{supp} f \subset \mathcal{O}_1 \cap \mathcal{O}$. Hence this trajectory is a fast solution of X_{f_n}, and the function f is fast. ∎

If $\boldsymbol{r} = (r_j)_{j\in\mathbb{N}^*}$ is a sequence of $\mathbb{R}_+^* \cup \{+\infty\}$ with $0 < r = \inf_{j\in\mathbb{N}^*} r_j < +\infty$, we define

$$D(\boldsymbol{r}) = \left\{ z = \sum_{j=1}^{+\infty} p_j\varphi_j^+ + q_j\varphi_j^- \Big/ \forall j \in \mathbb{N},\ p_j^2 + q_j^2 < r_j^2 \right\},$$

$$E(\boldsymbol{r}) = \left\{ z = \sum_{j=1}^{+\infty} p_j\varphi_j^+ + q_j\varphi_j^- \Big/ \sum_{j=1}^{+\infty} \frac{p_j^2 + q_j^2}{r_j^2} < 1 \right\}.$$

Remark that if $\boldsymbol{r} = (r, +\infty, \dots, +\infty)$, $D(\boldsymbol{r})$ is a symplectic cylinder $\mathcal{C}_{r,1}$.

Theorem 2.15. *We have* $c(E(\boldsymbol{r})) = c(D(\boldsymbol{r})) = \pi r^2$

Proof. We have to check the following inequalities

(1) $c(E(\boldsymbol{r})) \geq \pi r^2$
(2) $c(D(\boldsymbol{r})) \leq \pi r^2$

then we will conclude by proposition 2.12.

(1) It is sufficient to prove that $c(B_1) \geq \pi$ (then the result follows by proposition 2.12).

Define $m = \pi - \varepsilon$. Choose $f : [0,1] \to \mathbb{R}_+$ satisfying :

$$\begin{cases} 0 \leq f'(t) < \pi \text{ for } t \in [0,1] \\ f(t) = 0 \text{ for } t \text{ near } 0 \\ f(t) = m \text{ for } t \text{ near } 1 \end{cases}$$

Then, define $H(x) = f(\|x\|^2)$ for x in $B(1)$. H is m-admissible. We want to prove that H is not fast. Consider

$$H_n(x) = f\left(\sum_{j=1}^{n}(p_j^2 + q_j^2)\right), \quad \text{where } x = \sum_j (p_j\varphi_j^+ + q_j\varphi_j^-).$$

Using the variables $I_j = \frac{1}{2}(p_j^2 + q_j^2)$ and $\theta_j = \arctan\left(\frac{p_j}{q_j}\right)$ we observe that non-constant periodic solutions corresponding to this hamiltonian has a period $T > 1$. Hence X_{H_n} has no fast trajectory and H is not fast.

(2) Denote $\mathcal{O} = D(\boldsymbol{r})$. Pick $m > \pi r^2$ and f a m-admissible function in $\mathcal{O}$. Since $f^{-1}(0)$ is not empty, there exists n such that $f^{-1}(0) \cap Z^n \neq \emptyset$. Denote $f_n = f|_{\mathcal{O}^n}$. Since $\partial\mathcal{O}^n \subset \partial\mathcal{O}$, we deduce that f_n equals m on a neighbourhood of $\partial\mathcal{O}^n$. Hence f_n is m-admissible.

Since $c_{2n}(\mathcal{O}^n) = \pi \min\limits_{1\le j\le n} r_j^2$, we have

$$c_{2n}(\mathcal{O}^n) \underset{n\to+\infty}{\longrightarrow} \pi \inf_{j\ge 1} r_j^2 = \pi r^2 < m.$$

Hence, for n sufficiently large $c_{2n}(\mathcal{O}^n) < m$. Therefore X_{f_n} has a fast periodic trajectory and the function f is fast. ■

Corollary 2.16. *We have $c(B_r) = c(\mathcal{C}_{r,1}) = \pi r^2$, and for each bounded open set $\mathcal{O}$ of Z we have $0 < c(\mathcal{O}) < +\infty$.*

The essential property of the capacity c is its invariance with respect to the flow maps of PDEs satisfying assumptions (H1), (H2) and (H3). In fact the non-squeezing theorem 2.1 is a consequence of the following result.

Theorem 2.17. *Let Φ_T the flow of an equation (2) satisfying the assumptions (H1), (H2) and (H3). For any open set $\mathcal{O}$ of Z we have*

$$c(\Phi_T(\mathcal{O})) = c(\mathcal{O}).$$

Proof. Let us denote $\Phi = \Phi_T$ and $\mathcal{Q} = \Phi(\mathcal{O})$. One easily checks that Φ^{-1} satisfies (H1), (H2) and (H3), therefore it is sufficient to prove that $c(\mathcal{Q}) \le c(\mathcal{O})$.

Take any $m > c(\mathcal{O})$ and any f m-admissible in $\mathcal{Q}$. We want to prove that f is fast.

Since f is m-admissible there exists $R > 0$ such that supp$f \subset B_R$. Define $R_1 = R + d(f)$, $\mathcal{Q}' = \mathcal{Q} \cap B_{R'}$ and $\mathcal{O}' = \Phi^{-1}(\mathcal{Q}')$. By assumption $\mathcal{O}'$ is bounded, hence there exists R' such that $\mathcal{O}' \subset B_{R'}$. Moreover we clearly have $\mathcal{O}' \subset \mathcal{O}$, thus by proposition 2.12

$$c(\mathcal{O}') \le c(\mathcal{O}). \tag{9}$$

We apply lemma 2.3 with N so large that $\varepsilon < \frac{1}{2}d(f)$, and we use the notations of the lemma 2.3 : $\Phi = e^{TJA}(I + \widetilde{\Phi}_\varepsilon)(I + \widetilde{\Phi}_N)$. We denote $\mathcal{O}_1$ and $\mathcal{O}_2$ the intermediate domains which arrise from the decomposition

$$\mathcal{O}' \xrightarrow{I+\widetilde{\Phi}_N} \mathcal{O}_1 \xrightarrow{I+\widetilde{\Phi}_\varepsilon} \mathcal{O}_2 \xrightarrow{e^{TJA}} \mathcal{Q}'.$$

We also denote

$$f_2 = \left(f \circ e^{TJA}\right)\big|_{\mathcal{O}_2}.$$

Observe that f_2 is m-admissible on $\mathcal{O}_2$. Indeed f is m-admissible on $\mathcal{Q}$ and also on $\mathcal{Q}'$ (by definition of $\mathcal{Q}'$). Since e^{tJA} is an isometry, f_2 is m-admissible.

Then, we extend f_2 as m outside $\mathcal{O}_2$, and we denote $\widetilde{f}$ its restriction to $\mathcal{O}_1$. By (4) the ε-neighbourhood of $\partial\mathcal{O}_1$ is contained in the 2ε-neighbourhood of $\partial\mathcal{O}_2$. Since $\varepsilon < \frac{1}{2}d(f)$, we deduce that $\widetilde{f}$ equals m on a neighbourhood of $\partial\mathcal{O}_1$. Moreover $\widetilde{f}^{-1}(0) = f_2^{-1}(0) \subset \mathcal{O}_1 \cap \mathcal{O}_2$. Indeed by remark 2.8

$$\text{dist}(f_2^{-1}(0), \partial\mathcal{O}_2) \ge d(f)$$

$$\text{and } \text{dist}(\partial\mathcal{O}_1, \partial\mathcal{O}_2) \le \frac{1}{2}d(f).$$

Hence $\widetilde{f}$ is m-admissible on $\mathcal{O}_1$.

Using lemma 2.13 and (9), we deduce that

$$c(\mathcal{O}_1) = c\left((I + \widetilde{\Phi}_N)(\mathcal{O}')\right) = c(\mathcal{O}') \leq c(\mathcal{O}) < m.$$

Hence $\widetilde{f}$ is m-admissible on $\mathcal{O}_1$ and $c(\mathcal{O}_1) < m$, thus $\widetilde{f}$ is fast. So for n sufficiently large, the vectorfield $X_{\widetilde{f}_n}$ (where $\widetilde{f}_n = \widetilde{f}|_{\mathcal{O}_1^n}$) has a fast solution. By lemma 2.10 this solution lies in supp$\widetilde{f}$ and by remark 2.8 supp$\widetilde{f}$ =suppf_2, so this solution is also a fast solution of $X_{f_2^n}$ (where $f_2^n = f_2|_{\mathcal{O}_2^n}$). Hence f_2 is fast too. Finally f is also fast ($f_2 = \left(f \circ e^{TJA}\right)\big|_{\mathcal{O}_2}$). ■

3. Application to the BBM equation

In this section we prove that the BBM equation

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{T} \\ u(0,x) = u_0(x) \end{cases} \tag{10}$$

is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$ (we will follow the proof given in [2] for $x \in \mathbb{R}$) and has the non-squeezing property (theorem 1.1).

3.1. **Bilinear estimates.** We start by two helpful inequalities.
Let $\varphi(k) = \frac{k}{1+k^2}$ and $\varphi(D)$ the Fourier multiplier operator defined by $\widehat{\varphi(D)u}(k) = \varphi(k)\widehat{u}(k)$.

Lemma 3.1. *Let $u \in H^r(\mathbb{T})$ and $v \in H^{r'}(\mathbb{T})$ with $0 \leq r \leq s$, $0 \leq r' \leq s$ and $0 \leq 2s - r - r' < 1/4$. Then*

$$\|\varphi(D)(uv)\|_{H^s} \leq C_{r,r',s} \|u\|_{H^r} \|v\|_{H^{r'}}$$

Proof. We want to prove

$$\left\| \langle k \rangle^s \frac{k}{1+k^2} \widehat{uv}(k) \right\|_{\ell^2_k} \leq C \|u\|_{H^r} \|v\|_{H^{r'}}.$$

By duality it is sufficient to prove

$$\left\langle \langle k \rangle^s \frac{k}{1+k^2} \widehat{uv}, \widehat{w} \right\rangle_{\ell^2} \leq C \|u\|_{H^r} \|v\|_{H^{r'}} \|w\|_{L^2},$$

that is

$$I = \sum_{k \in \mathbb{Z}} k \langle k \rangle^{s-2} \widehat{uv}(k) \overline{\widehat{w}}(k) \leq C \|u\|_{H^r} \|v\|_{H^{r'}} \|w\|_{L^2}.$$

Let $f(k) = \langle k \rangle^r \widehat{u}(k)$, $g(k) = \langle k \rangle^{r'} \widehat{v}(k)$ and $h(k) = k \langle k \rangle^{-2(1+r+r'-2s)} \overline{\widehat{w}}(k)$. Since

$$\widehat{uv}(k) = \sum_{l \in \mathbb{Z}} \widehat{u}(l) \widehat{v}(k-l)$$

we have

$$I = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \frac{\langle k \rangle^{-3s+2r+2r'}}{\langle l \rangle^r \langle k-l \rangle^{r'}} f(l) g(k-l) h(k).$$

We have $-2s + r + r' \leq 0$ and $-s + r \leq 0$ and $-s + r' \leq 0$ so
$-3s + 2r + 2r' = -2s + r + r' + (-s + r') + r \leq r$ and $-3s + 2r + 2r' \leq r'$.

Hence $\dfrac{\langle k\rangle^{-3s+2r+2r'}}{\langle l\rangle^{r}\langle k-l\rangle^{r'}}$ is bounded for k and l in $\mathbb{Z}$. Then (by Cauchy-Schwarz inequality and Young's inequality)

$$\begin{aligned}
I &\lesssim \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} f(l)g(k-l)h(k)\\
&\lesssim \|f\|_{\ell^2}\,\|g*h(-\cdot)\|_{\ell^2}\\
&\lesssim \|f\|_{\ell^2}\,\|g\|_{\ell^2}\,\|h\|_{\ell^1}\\
&\lesssim \|u\|_{H^r}\,\|v\|_{H^{r'}}\,\|w\|_{L^2}\left\|\frac{k}{(1+k^2)^{1+r+r'-2s}}\right\|_{\ell^2_k}.
\end{aligned}$$

Since $2s-r-r'<1/4$ we have $1+r+r'-2s>3/4$. Hence

$$\left\|\frac{k}{(1+k^2)^{1+r+r'-2s}}\right\|_{\ell^2_k}<+\infty.$$

■

In subsection 3.3 we will use this lemma in the particular case $r=r'=s\geq 0$, that is

$$\|\varphi(D)(uv)\|_{H^s}\leq C_s\,\|u\|_{H^s}\,\|v\|_{H^s}$$

whereas in subsection 3.4 and 3.5 we will need the general case $0\leq r,r'<s$.

Lemma 3.2. *Let $u\in H^r(\mathbb{T})$ and $v\in H^s(\mathbb{T})$ with $0\leq s\leq r$ and $r>\frac{1}{2}$, then*

$$\|\varphi(D)(uv)\|_{H^{s+1}}\leq C\,\|u\|_{H^r}\,\|v\|_{H^s}.$$

Proof. Since $r>\frac{1}{2}$ and $r\geq s\geq 0$, the elements of $H^r(\mathbb{T})$ are multipliers in $H^s(\mathbb{T})$, which is to say

$$\|uv\|_{H^s}\lesssim \|u\|_{H^r}\,\|v\|_{H^s}.$$

Hence

$$\begin{aligned}
\|\varphi(D)(uv)\|_{H^{s+1}} &= \left\|\frac{\langle k\rangle^{s+1}k}{\langle k\rangle^{2}}\widehat{uv}\right\|_{\ell^2_k}\\
&\leq \|\langle k\rangle^{s}\,\widehat{uv}\|_{\ell^2_k}\\
&= \|uv\|_{H^s}\\
&\lesssim \|u\|_{H^r}\,\|v\|_{H^s}.
\end{aligned}$$

■

3.2. **Hamiltonian formalism for BBM equation.** Recall that BBM equation reads

$$u_t+u_x+uu_x-u_{txx}=0.$$

Let us prove that BBM equation is a hamiltonian equation (2).

First BBM can be written

$$u_t=-\partial_x(1-\partial_x^2)^{-1}(u+\frac{u^2}{2}).$$

Denote $Z = H_0^{1/2}(\mathbb{T}) = \left\{u \in H^{1/2} / \int_{\mathbb{T}} u = 0\right\}$ with the following norm

$$\|u\|_Z = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1+k^2}{k}(a_k^2 + b_k^2)$$

where a_k and b_k are the (real) Fourier coefficients of u.

Consider the Hilbert basis of Z given by

$$\varphi_n^+(x) = \sqrt{\frac{n}{\pi(n^2+1)}}\cos(nx), \qquad \varphi_n^-(x) = \sqrt{\frac{n}{\pi(n^2+1)}}\sin(nx).$$

We have $Z_+ = H_0^{1/2+\varepsilon} < H_0^{1/2} < H_0^{1/2-\varepsilon} = Z_-$, where $\varepsilon > 0$ will be fixed later.

Define

$$H(u) = \int_{\mathbb{T}} \left(\frac{u(x)^2}{2} + \frac{u(x)^3}{6}\right) dx,$$

we have

$$\nabla_{L^2} H(u) = u + \frac{u^2}{2}.$$

Assume

$$u(t) = \sum_n p_n(t)\varphi_n^+ + q_n(t)\varphi_n^-$$

and

$$\nabla_{L^2} H(u) = \sum_n \alpha_n\varphi_n^+ + \beta_n\varphi_n^-.$$

Denoting $\widetilde{H}(p,q) = H(\sum_n p_n(t)\varphi_n^+ + q_n(t)\varphi_n^-)$ we deduce that

$$\frac{\partial\widetilde{H}}{\partial p_n} = \left\langle \nabla_{L^2} H(u), \varphi_n^+\right\rangle_{L^2} = \alpha_n \left\|\varphi_n^+\right\|_{L^2}^2 = \frac{n\alpha_n}{1+n^2}$$

and

$$\frac{\partial\widetilde{H}}{\partial q_n} = \frac{n\beta_n}{1+n^2}.$$

Hence

$$\begin{aligned}\dot{u} = \sum_n \dot{p}_n\varphi_n^+ + \dot{q}_n\varphi_n^- &= (1-\partial_x^2)^{-1}\partial_x(-\nabla_{L^2} H(u)) \\ &= \sum_n \frac{-n\alpha_n}{1+n^2}\varphi_n^- + \frac{n\beta_n}{1+n^2}\varphi_n^+\end{aligned}$$

so

$$\begin{cases} \dot{p}_n = \dfrac{n\beta_n}{1+n^2} = \dfrac{\partial\widetilde{H}}{\partial q_n} \\ \dot{q}_n = \dfrac{-n\alpha_n}{1+n^2} = -\dfrac{\partial\widetilde{H}}{\partial p_n} \end{cases}$$

That is $\dot{u} = J\nabla_Z H(u)$.

3.3. **Verification of (H1).**

3.3.1. *Local well-posedness.* Recall that $\varphi(k) = \frac{k}{1+k^2}$, the equation (10) can be written in the form :

$$\begin{cases} iu_t = \varphi(D)u + \frac{1}{2}\varphi(D)u^2 \\ u(0,x) = u_0(x) \end{cases} \tag{11}$$

Let $e^{-it\varphi(D)}$ be the unitary group defining the associated free evolution. That is, $e^{-it\varphi(D)}u_0$ solves the Cauchy problem

$$\begin{cases} iu_t = \varphi(D)u \\ u(0,x) = u_0(x) \end{cases} \tag{12}$$

Then, (11) may be rewritten as the integral equation

$$u(t) = e^{-it\varphi(D)}u_0 - \frac{i}{2}\int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)^2)d\tau = \mathcal{A}(u)(t,\cdot).$$

Let $X_T^s = C^0([-T,T], H^s(\mathbb{T}))$. The H^s norm is clearly preserved by the free evolution, thus

$$\left\| e^{-it\varphi(D)}u_0 \right\|_{X_T^s} = \|u\|_{H^s}. \tag{13}$$

Theorem 3.3. *Let $s \geq 0$. For any $u_0 \in H^s(\mathbb{T})$, there exist a time T (depending on u_0) and a unique solution $u \in X_T^s$ of (10). The maximal existence time T_s has the property that*

$$T_s \geq \frac{1}{4C_s\|u_0\|_{H^s}}$$

with C_s the constant from lemma 3.1 (in the special case $r = r' = s$).

Moreover, for $R > 0$, let T denote a uniform existence time for (10) with $u_0 \in B_R(H^s(\mathbb{T}))$, then the map $\Phi : u_0 \mapsto u$ is real-analytic from $B_R(H^s(\mathbb{T}))$ to X_T^s.

Proof. Let $R = 2\|u_0\|_{H^s}$. For any $u \in B_R(X_T^s)$, by (13) and lemma 3.1 (with $r = r' = s$) we have

$$\begin{aligned} \|\mathcal{A}(u)\|_{X_T^s} &\leq \left\| e^{-it\varphi(D)}u_0 \right\|_{X_T^s} + \frac{1}{2}\left\| \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(u(\tau)^2)d\tau \right\|_{X_T^s} \\ &\leq \|u_0\|_{H^s} + \frac{C_sT}{2}\|u\|_{X_T^s}^2 \\ &\leq \|u_0\|_{H^s} + \frac{C_sT}{2}R^2 \\ &\leq R \qquad \text{for } T = \frac{2}{C_sR} \end{aligned}$$

and for any $u, v \in B_R(X_T^s)$, by lemma 3.1 (with $r = r' = s$) we have

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{X_T^s} \leq \frac{C_sT}{2}\|u-v\|_{X_T^s}\|u+v\|_{X_T^s} \leq C_sTR\|u-v\|_{X_T^s}.$$

Hence, $\mathcal{A}$ is a contraction mapping of $B_R(X_T^s)$ for $T = \frac{1}{2C_sR} = \frac{1}{4C_s\|u_0\|_{H^s}}$. Thus $\mathcal{A}$ has a unique fixed point which is a solution of (10) on time interval $[-T,T]$.

Let us consider now the smoothness of Φ. Let $\Lambda : H^s(\mathbb{T}) \times X_T^s \longrightarrow X_T^s$ be defined as

$$\Lambda(u_0, v)(t) = v(t) - e^{-it\varphi(D)}u_0 - \frac{i}{2}\int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(v(\tau)^2)d\tau.$$

Due to lemme 3.1 (with $r = r' = s$), Λ is a smooth map from $H^s(\mathbb{T}) \times X_T^s$ to X_T^s. Let $u \in X_T^s$ be the solution of (10) with initial data $u_0 \in H^s(\mathbb{T})$, which is to say $\Lambda(u_0, u) = 0$. Thus, the Fréchet derivative of Λ with respect to the second variable is the linear map :

$$\Lambda'(u_0, u)(t)[h] = h - \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)h(\tau))d\tau.$$

Still by lemma 3.1 we get

$$\left\| \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)h(\tau))d\tau \right\|_{X_T^s} \leq CT \left\| u \right\|_{H^s} \left\| h \right\|_{H^s}.$$

So, for T' sufficiently small (depending only on $\|u\|_{H^s}$), $\Lambda'(u_0, u)(t)$ is invertible since it is of the form $Id + K$ with

$$\|K\|_{\mathcal{B}(X_{T'}^s, X_{T'}^s)} < 1$$

where $\mathcal{B}(X_{T'}^s, X_{T'}^s)$ is the Banach space of bounded linear operators on $X_{T'}^s$. Thus $\Phi : B_R(H^s(\mathbb{T})) \to X_T^s$ is real-analytic by Implicit Function Theorem. ■

3.3.2. *Global well-posedness.*

Theorem 3.4. *The solution defined in theorem 3.3 is global in time.*

Proof. Fix $T > 0$. The aim is to show that corresponding to any initial data $u_0 \in H^s$, there is a unique solution of (10) that lies in X_T^s. Because of theorem 3.3, this result is clear for data that is small enough in H^s, and it is sufficient to prove the existence of a solution corresponding to initial data of arbitrary size (uniqueness is a local issue). Fix $u_0 \in H^s$ and let N be such that

$$\sum_{|k| \geq N} \langle k \rangle^{2s} \left| \widehat{u_0}(k) \right|^2 \leq T^{-2}.$$

Such values of N exist since $\langle k \rangle^s \left| \widehat{u_0}(k) \right|$ is in ℓ^2. Define

$$v_0(x) = \sum_{|k| \geq N} e^{ixk}\widehat{u_0}(k).$$

By theorem 3.3, there exists a unique $v \in X_T^s$ solution of (10) with initial data v_0. Split the initial data u_0 into two pieces: $u_0 = v_0 + w_0$; and consider the following Cauchy problem (where v is now fixed)

$$\begin{cases} w_t - w_{xxt} + w_x + ww_x + (vw)_x \\ w(0,x) = w_0(x) \end{cases} \tag{14}$$

If there exists a solution w of (14) in X_T^s then $v + w$ will be a solution of (10) in X_T^s.

First, w_0 is in $H^r(\mathbb{T})$ for all $r > 0$, in particular $w_0 \in H^1(\mathbb{T})$. And (14) may be rewritten as the integral equation

$$w(t,x) = e^{-it\varphi(D)}w_0 - \frac{i}{2}\int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(vw + w^2)d\tau = \mathcal{K}(w).$$

This problem can be solved locally in time on $H^1(\mathbb{T})$ by the same arguments used to prove theorem 3.3. Indeed for any $w \in B_R(X^1_S)$, by lemma 3.2 (with $r = 1$ and $s = 0$) and lemma 3.1 (with $r = r' = s = 1$)

$$\begin{aligned} \|\mathcal{K}(w)\|_{X^1_S} &\le \|w_0\|_{H^1} + CS\left(\|v\|_{X^0_S}\|w\|_{X^1_S} + \|w\|^2_{X^1_S}\right) \\ &\le CS\|v\|_{X^0_S} R \end{aligned} \tag{15}$$

and for any w_1 and w_2 in $B_R(X^1_S)$

$$\begin{aligned} &\|\mathcal{K}(w_1) - \mathcal{K}(w_2)\|_{X^1_S} \\ &\qquad \le CS\left(\|v\|_{X^0_S}\|w_1 - w_2\|_{X^1_S} + \|w_1 - w_2\|_{X^1_S}\|w_1 + w_2\|_{X^1_S}\right) \\ &\qquad \le CS\left(\|v\|_{X^0_S} + 2R\right)\|w_1 - w_2\|_{X^1_S}. \end{aligned} \tag{16}$$

Hence, by (15) and (16), $\mathcal{K}$ has a unique fixed point in X^1_S. Therefore we have a solution w in X^1_S for a small time S.

If we have an *a priori* bound on the H^1-norm of w showing it was bounded on the interval $[-T, T]$ it would follow that a solution on $[-T, T]$ could be obtained.

The formal steps of this inequality are as folllows (the justification is made by regularizing). Multiply the equation (14) by w, integrate over $\mathbb{T}$, and after integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}\left(w(t,x)^2 + w_x(t,x)^2\right)dx - \int_{\mathbb{T}} v(t,x)w(t,x)w_x(t,x)dx = 0.$$

By Hölder and Sobolev inequalities we deduce

$$\begin{aligned} \left|\int_{\mathbb{T}} v(t,x)w(t,x)w_x(t,x)dx\right| &\le \|v(t,\cdot)\|_{L^2}\|w(t,\cdot)\|_{L^\infty}\|w_x(t,\cdot)\|_{L^2} \\ &\le C\|v(t,\cdot)\|_{L^2}\|w(t,\cdot)\|^2_{H^1}. \end{aligned}$$

Hence

$$\frac{d}{dt}\|w(t,\cdot)\|^2_{H^1} \le 2C\|v(t,\cdot)\|_{L^2}\|w(t,\cdot)\|^2_{H^1}$$

and by Gronwall's inequality

$$\|w(t,\cdot)\|_{H^1} \le \|w_0\|_{H^1}\exp\left(C\int_0^t \|v(\tau,\cdot)\|_{L^2}d\tau\right).$$

We deduce from this *a priori* bound that the solution w of (14) exists on the interval $[-T, T]$, and $v + w$ is a solution of (10) in X^s_T. ■

3.4. **Verification of (H2).**

Proposition 3.5. *For any $T > 0$, $R > 0$, and $s > 0$ there exists R' such that*

$$\forall 0 \leq t \leq T, \Phi_t(B_R(H^s)) \subset B_{R'}(H^s).$$

With $s = \frac{1}{2}$ we deduce that Φ satisfies (H2).

Proof. The result is clear for $s \geq 1$, so we assume that $0 < s < 1$. Fix $T > 0$, $R > 0$ and u_0 in H^s such that $\|u_0\|_{H^s} \leq R$. Using the same idea as in theorem 3.4 split u_0 into two pieces $u_0 = v_0 + w_0$, where

$$v_0 = \sum_{|k| \geq N} \widehat{u_0}(k) e^{ikx}.$$

Using the same notations, let v be the solution of BBM equation with the initial data v_0 and w the solution of (14). We want to control v and w in H^s-norm.

Fix $\varepsilon > 0$ such that $\varepsilon < 1/8$ and $s - \varepsilon > 0$, we have

$$\|v_0\|_{H^{s-\varepsilon}} \leq N^{-\varepsilon} \|v_0\|_{H^s}.$$

We choose $N = \left(\frac{4RC}{T}\right)^{1/\varepsilon}$ where C is the constant of lemma 3.1. Hence we have

$$\|v_0\|_{H^{s-\varepsilon}} \leq \frac{1}{4CT} = M.$$

By local theory (theorem 3.3) the flow map

$$\Phi : B_M(H^{s-\varepsilon}) \longrightarrow X_T^{s-\varepsilon}$$

is continuous. Since $H^s \cap B_M(H^{s-\varepsilon})$ is precompact in $B_M(H^{s-\varepsilon})$ we have

$$\sup_{v_0 \in H^s \cap B_M(H^{s-\varepsilon})} \|\Phi(v_0)\|_{X^{s-\varepsilon}} = C_1(R, T).$$

By lemma 3.1 with $r = r' = s - \varepsilon$ we have

$$\|v\|_{X^s} \leq \|v_0\|_{H^s} + CT \|v\|_{X^{s-\varepsilon}}^2 \leq R + CTC_1(R,T)^2 = C_2(R,T).$$

The *a priori* bound on w gives

$$\begin{aligned} \|w(t)\|_{H^s} \leq \|w(t)\|_{H^1} &\leq \|w_0\|_{H^1} \exp\left(C \int_0^t \|v(\tau, \cdot)\|_{L^2} \, d\tau\right) \\ &\leq N^{1-s} \|w_0\|_{H^s} e^{CTC_2(R,T)} \\ &\leq C_3(R,T). \end{aligned}$$

Hence, we have

$$\|u\|_{X_T^s} \leq C_2(R,T) + C_3(R,T)$$

■

Corollary 3.6. *For each $T > 0$ and $s > 0$, the flow map $\Phi : H^s \to X_T^s$ is real analytic.*

Proof. Let $u_0 \in H^s$, $R = \|u_0\|_{H^s}$ and $T > 0$. By proposition 3.5, there exists R' such that $\Phi_t(B_{2R}(H^s)) \subset B_{R'}(H^s)$, for all $t \in [0, T]$. And by local theory (theorem 3.3) there exists a small time τ such that $\Phi : B_{R'}(H^s) \to X_\tau^s$ is real analytic. Splitting the time intervalle $[0, T]$ into $\bigcup [k\tau, (k+1)\tau]$, we deduce that $\Phi : H^s \to X_T^s$ is real analytic. ■

3.5. **Verification of (H3).** Recalll that $\widetilde{\Phi}$ denote the non-linear part of the flow, that is $\Phi_t = e^{-it\varphi(D)}(I + \widetilde{\Phi}_t)$. The assumption (H3) results from

Proposition 3.7. *For any $u_0, v_0 \in B_R(H^{1/2}(\mathbb{T}))$ we have the following estimate*

$$\left\|\widetilde{\Phi}(u_0) - \widetilde{\Phi}(v_0)\right\|_{X_T^{1/2+\varepsilon}} \leq C_{R,T,\varepsilon} \left\|u_0 - v_0\right\|_{H^{1/2-\varepsilon}}$$

for $0 < \varepsilon < 1/12$.

Proof. Let $0 < \varepsilon < \frac{1}{12}$, u_0 and v_0 in $B_R(H^{1/2})$. Denoting u and v the solutions of BBM equation with initial data u_0 and v_0. By lemma 3.1 with $s = \frac{1}{2} + \varepsilon$ and $r = \frac{1}{2}$ and $r' = \frac{1}{2} - \varepsilon$ and (H2) we have

$$\begin{aligned}\left\|\widetilde{\Phi}_t(u_0) - \widetilde{\Phi}_t(v_0)\right\|_{X_T^{1/2+\varepsilon}} &\leq CT \left\|u+v\right\|_{X_T^{1/2}} \left\|u-v\right\|_{X_T^{1/2-\varepsilon}} \\ &\leq 2CTR'_{R,T} \left\|u-v\right\|_{X_T^{1/2-\varepsilon}} .\end{aligned}$$

Since u_0 and v_0 are in $B_R(H^{1/2})$ and Φ is C^1 on $B_R(H^{1/2})$ which is a relatively compact subset of $H^{1/2-\varepsilon}$ we have

$$\begin{aligned}\left\|u-v\right\|_{X_T^{1/2-\varepsilon}} &= \left\|\Phi_t(u_0) - \Phi_t(v_0)\right\|_{X_T^{1/2-\varepsilon}} \\ &\leq \sup_{w_0 \in B_R(H^{1/2}) \cap H^{1/2-\varepsilon}} \left(\left\|d\Phi(w_0)\right\|_{\mathcal{B}\left(H^{1/2-\varepsilon}, X_T^{1/2-\varepsilon}\right)}\right) \left\|u_0 - v_0\right\|_{H^{1/2-\varepsilon}} \\ &\leq C_{R,T,\varepsilon} \left\|u_0 - v_0\right\|_{H^{1/2-\varepsilon}} .\end{aligned}$$

■

Hence, we can apply the non-squeezing theorem (theorem 2.1) and that proves the theorem 1.1.

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